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Gain-Scheduling Stability Issues Using Differential Inclusion and Fuzzy Systems

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I. Introduction

SINCE its first appearance, the Takagi–Sugeno (TS) fuzzy model theory[1] has proven useful in the description of nonlinear dynamic systems as a means of blending of models obtained by local analysis. Such descriptions are referred to as model-based fuzzy systems (MBFS). In addition, the TS approach can be used for the synthesis of fuzzy gain-scheduled controllers. The stability of MBFS was studied by Hallendorn et al.,[2,3] who defined a stability test by imposing some conditions on the local control laws. The present work describes a new stability criterion, which relaxes the bounds in Ref. 3, yielding a less conservative condition. Two case studies are presented comparing the use of off-equilibrium vs equilibrium grid points and fuzzy vs crisp scheduling.

II. Modeling and Control

Consider a nonlinear continuous and continuously differentiable system of the form

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}), \qquad \bar{y} = \bar{x} \tag{1}$$

where $\bar{x} \in \Re^n$, $\bar{u} \in \Re^m$, and $f : \Re^n \times \Re^m \to \Re^n$. We wish to design a controller capable of following some desired trajectory $(\bar{x}_r, \bar{u}_r)$, where $\bar{x}_r$ is a differentiable, slowly varying state trajectory and $\bar{u}_r$ is the nominal input necessary to follow the unperturbed $\bar{x}_r$ state.[2,3] Let us define a subset $XU \subset \Re^{n+m}$ of the system's state and input spaces as a bound on all of the possible state and input values. Let us also define a set of operating points as $(x_i, u_i) \in XU$, $i \in I$ with I set of all positive integers that form a regular (or irregular) grid J in the trajectory space. Linearization of Eq. (1) about all of the points in J yields

$$A_i = \left.\frac{\partial f}{\partial x}\right|_{(x_i,u_i)}, \qquad B_i = \left.\frac{\partial f}{\partial u}\right|_{(x_i,u_i)} \tag{2}$$

resulting in perturbed dynamics about the linearization points given by

$$\dot{\bar{x}} = A_i(\bar{x} - \bar{x}_i) + B_i(\bar{u} - \bar{u}_i) + f(\bar{x}_i, \bar{u}_i) = A_i\bar{x} + B_i\bar{u} + \bar{d}_i$$
$$\bar{d}_i = f(\bar{x}_i, \bar{u}_i) - A_i\bar{x}_i - B_i\bar{u}_i \tag{3}$$

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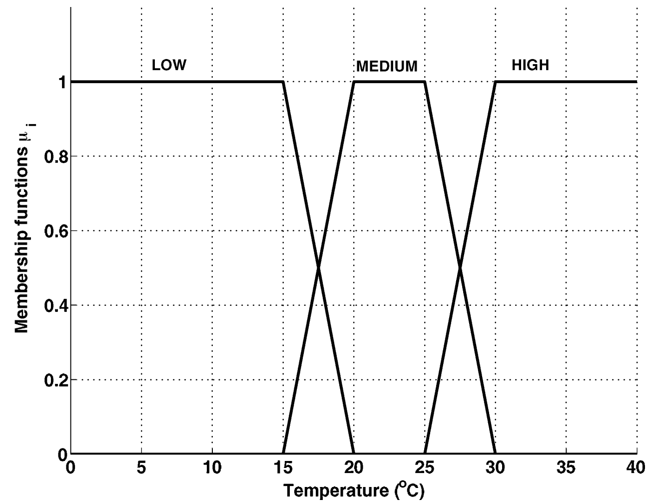

Fig. 1 Example of fuzzy rule set.

When the linearized systems (3) are interpolated through a TS model, a nonlinear approximation of Eq. (1) is obtained, given by

$$\dot{\bar{x}} \cong \hat{f}(\bar{x}, \bar{u}) = \sum_{i \in I} \mu_i(\bar{x}, \bar{u}) \cdot (A_i\bar{x} + B_i\bar{u} + \bar{d}_i) \tag{4}$$

A fuzzy control law for system (4) is also designed as a gain-scheduling controller based on a TS model. Under the hypothesis of controllability for all $(A_i, B_i)\forall i \in I$ and being all of the states measured, full-state feedback linear control laws can be synthesized and interpolated through a fuzzy TS system yielding

$$\bar{u} = \bar{u}_r + \sum_{j \in I} \upsilon_j(\bar{x}, \bar{u}) \cdot K_j[(\bar{x} - \bar{x}_j) - (\bar{x}_r - \bar{x}_j)]$$
$$= \bar{u}_r + \sum_{j \in I} \upsilon_j(\bar{x}, \bar{u}) \cdot K_j[\bar{x} - \bar{x}_r] \tag{5}$$

In Eqs. (4) and (5) the expressions μ_i, υ_j represent the TS linear membership functions relating the input variables to the fuzzy domain described by IF-THEN-ELSE rules consequent. The fuzzy system membership functions are chosen such as they constitute a convex sum over the input range XU. An example is shown in Fig. 1.

Substituting Eq. (5) in Eq. (4), the closed-loop perturbed system dynamics become

$$\dot{\bar{x}} - \dot{\bar{x}}_r = \sum_i \mu_i(\bar{x}, \bar{u}) \cdot \left\{A_i + B_i\left[\sum_j \upsilon_j(\bar{x}, \bar{u})K_j\right]\right\}(\bar{x} - \bar{x}_r) + \varepsilon$$
$$\varepsilon = \sum_i \mu_i(\bar{x}, \bar{u}) \cdot (A_i\bar{x}_r + B_i\bar{u}_r + \bar{d}_i) - \dot{\bar{x}}_r \tag{6}$$

Note that the term $\sum_i A_i\bar{x}_r$ is added and subtracted so that the matrix

$$\sum_i \mu_i(\bar{x}, \bar{u}) \cdot \left\{A_i + B_i\left[\sum_j \upsilon_j(\bar{x}, \bar{u})K_j\right]\right\} \tag{7}$$

gives the dynamics of the perturbation from the desired trajectory. Also, from the definition of $\bar{d}_i\varepsilon$ represents the error with respect to $\bar{x}_r$ as a result of the approximation of f with the TS model.

III. Stability Analysis

Let us now derive the asymptotic stability conditions of the TS fuzzy gain-scheduling controller around $(\bar{x}_r, \bar{u}_r)$.

Definition: Given the grid point set J and any linearized dynamics (A_i, B_i), $i \in J$, J_i is defined as the set of all indexes m of the neighbourhood points of $(\bar{x}_i, \bar{u}_i)$, whose controllers K_m have a non-negligible influence over (A_i, B_i).

From the preceding, J_i contains all points m such that $\upsilon_m(\bar{x},\bar{u})>0, \forall(\bar{x},\bar{u})\in\{(\bar{x},\bar{u}):\mu_i(\bar{x},\bar{u})>0\}$. Given a generic input state pair $(\bar{x}_l,\bar{u}_l)$, $l\notin I$, the stability property for the tracking error $(\bar{x}_l-\bar{x}_r)\to 0$ requires that the following conditions be satisfied.

Condition 1: Suppose $(\bar{x}_i,\bar{u}_i)\in J$ is the nearest linearization grid point to the operating point $(\bar{x}_l,\bar{u}_l)\in J$. The system (A_i, B_i) remains closed-loop stable using a convex combination of controllers K_m, $m\in J_i$.

Condition 1 is verified using the following test, which is based on differential inclusion theory[4,5]; the test guarantees that the TS modelling of the plant $\widehat{f}(\cdot)$ is stable when controlled by the TS fuzzy controller for all possible controller combinations. Consider the closed-loop system dynamics about $(\bar{x}_i,\bar{u}_i)$:

$$\dot{\bar{x}}=\left\{A_i+B_i\left[\sum_j\upsilon_j(\bar{x},\bar{u})\cdot K_j\right]\right\}\bar{x}$$

$$=\sum_j\upsilon_j(\bar{x},\bar{u})\cdot(A_i+B_iK_j)\bar{x} \tag{8}$$

obtained by a convex combination of controllers K_m, $m\in J_i$. Equation (8) has the form of a polytopic differential inclusion, where the vertices are the matrices $(A_i+B_iK_j).j\in J_i$. Differential inclusion theory states that closed-loop stability for the vertices of a polytope yields stability of the whole convex combination; therefore, stability of all $(A_i+B_iK_j)$, $\forall j\in J_i$ is required. The stability test is repeated for all grid points obtaining

$$\forall i\in I,\qquad \forall j\in J_i$$

$$\exists P_i>0:(A_i+B_iK_j)^TP_i+P_i(A_i+B_iK_j)<0 \tag{9}$$

Inequality (9) can be easily solved using linear-matrix-inequalities (LMI) techniques. If the LMI test fails, then the grid J must be made denser. Furthermore, the LMI test suggests where to add additional linearization points, in order to make the closed-loop system stable. The proposed stability test improves that of Ref. 5 because it does not require that the closed-loop eigenvalues be the same for all operating points $i\in I$ [see Ref. 5, Eq. (23)].

Condition 2: The approximation error caused by linearization and successive TS fuzzy modeling with respect to the original nonlinear system is small enough as not to compromise robust stability with respect to structured uncertainties.

Let us suppose that the desired closed-loop dynamics are given by

$$A_d=A_i+B_iK_i,\qquad \forall i\in I \tag{10}$$

then, from Eq. (8)

$$\sum_j\upsilon_j(\bar{x},\bar{u})\cdot(A_i+B_iK_j)\bar{x}=A_d+\sum_j\upsilon_j(\bar{x},\bar{u})\delta A_{ij} \tag{11}$$

with

$$\delta A_{ij}=B(\bar{x}_i,\bar{u}_i)[K(\bar{x}_j,\bar{u}_j)-K(\bar{x}_i,\bar{u}_i)] \tag{12}$$

Hallendorn et al.[2] and Boyd et al.[5] propose to test the stability of

$$A+\sum_i\mu_i(\bar{x},\bar{u})\sum_j\upsilon_j(\bar{x},\bar{u})\cdot\delta A_{ij} \tag{13}$$

using the robust stability theorem under structured uncertainties found in Ref. 6. To satisfy condition 2, the uncertainty in the systems (A_i,B_i) caused by linearization errors is modeled by a set of possible parametric uncertainties. The maximum parametric variations for which stability is guaranteed can be computed using LMI techniques. If this result is larger than the maximum functional error, then the system remains stable in the entire convex combination of grid J points. The functional approximation error caused by linearization is computed as follows [7]:

$$\|f-\widehat{f}\|_\infty\le M/2[\Delta(J,XU)]^2=\varepsilon \tag{14}$$

where ε is the maximum approximation error and

$$\|\nabla^2 f(\bar{x},\bar{u})\|_\infty\le M,\qquad \forall(\bar{x},\bar{u})\in XU$$

$$\Delta(A,B)=\inf_{a\in A}\sup_{b\in B}\|a-b\|_2 \tag{15}$$

The term $\Delta(J,XU)$ represents a measure of the inverse grid point's density: a small Δ implies a dense grid. In fact, the particular choice for M gives a conservative estimate for ε. Some regions of XU might be more "regular," thus requiring a sparse grid. A local value ε_i for the error can be defined as

$$M_i/2(\Delta\{J_i,\text{conv}[(\bar{x}_k,\bar{u}_k),k\in J_i]\})^2=\varepsilon_i \tag{16}$$

$$\|\nabla^2 f(\bar{x},\bar{u})\|_\infty\le M_i,\qquad [\forall(\bar{x},\bar{u})\in\text{conv}[(\bar{x}_k,\bar{u}_k),k\in J_i] \tag{17}$$

where *conv*(.) indicates convex closure. Grid density can be adapted depending on the various M_i to keep ε_i below the desired approximation error ε. To test the robustness of the control system, let us define a set of structured uncertainties $E_{l,m}$ as matrices with 1 in the (l,m) position and zero otherwise. Define also the finite set

$$M_{Ji}=\left\{(l,m):(A_i+B_iK_j)_{(l,m)}\ne 0,\ j\in J_i\right\} \tag{17'}$$

and an index p to the elements in the set M_{Ji}. A matrix $E_{l,m}$ exists for each element p in M_{Ji}. Thus, the generic structured uncertainty can be modeled as $E_p=k_pE_{l,m}$, where the scalar k_p is the magnitude of uncertainty, and p the index in M_{Ji} corresponding to the given pair (l,m). Note that a matrix $E_{l,m}$ exists for each element in M_{Ji}. We can then solve the following LMI problem in the unknowns P_i and k_p:

$$\forall i,\forall j\in J_i\begin{cases}\rho_i=\max\sum_{p=1}^{|M_{Ji}|}|k_p|\\ P_i>0\\ (A_i+B_iK_j)^TP_i+P_i(A_i+B_iK_j)\\ \quad+\left(\sum_{p=1}^{|M_{Ji}|}E_p\right)^TP_i+P_i\left(\sum_{p=1}^{|M_{Ji}|}E_p\right)<0\end{cases} \tag{18}$$

yielding the bound ρ_i to the maximum allowable uncertainty.

If $\rho_i>\varepsilon_i$, $\forall i$, then system (1) with the fuzzy gain-scheduling controller given by Eq. (5) is stable for all trajectories in XU. If the stability test fails, it is necessary to select a denser grid. The values of ρ_i that break the ε_i threshold indicate which regions in XU require a denser grid.

Suppose now that Eq. (18) yields $\rho_i<\varepsilon_i$; we must find a new approximation error ε_i' such that $\rho_i'>\varepsilon_i'$. This implies that M_i and $\Delta(\cdot)$ must be reduced by increasing the number of grid points. The additional grid points must belong to $conv[(\bar{x}_k,\bar{u}_k),k\in J_i]$. The new grid J' satisfies therefore $J'\supset J$. The following now holds:

$$conv[(\bar{x}_k,\bar{u}_k),k\in J_i']\subset conv[(\bar{x}_k,\bar{u}_k),k\in J_i] \tag{19}$$

From Eq. (17) we have

$$M_i'\le M_i$$

$$\Delta\{J_i',conv[(\bar{x}_k,\bar{u}_k),k\in J_i']\}<\Delta\{J_i,conv[(\bar{x}_k,\bar{u}_k),k\in J_i]\} \tag{20}$$

So from Eq. (16) $\varepsilon_i' < \varepsilon_i$. LMI optimization is run again over J', yielding ρ_i'. The procedure is repeated until $\rho_i' > \varepsilon_i'$, $\forall i$.

IV. Case Studies

The first case study is a classical nonlinear system benchmark. Consider the system given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1^2 + x_2^2 + u \tag{21}$$

The objective of the control design is to follow reference trajectory given by a step command on the first state, that is, $\bar{x}_r = [A \;\; 0]^T$. A pole-placement method is used to assign the closed-loop linearized spectrum to (−2.5, −5.0). Two fuzzy controllers were designed:

1) The conventional controller case is obtained considering equilibrium points only. The equilibrium points for Eq. (21) belong to the manifold

$$\{(x_{1e}, x_{2e}, u_e) | x_{2e} = 0, u_e = -(x_{1e})^2\}$$

from which the following were selected: [(0.5,0,−0.25), (1.5,0,−2.25), (2.5,0,−6.25), (3.5,0,−12.25)].

2) For the Off-equilibrium case the following linearization points were selected: [(0.5,−1,0), (1.5,−1,0), (2.5,1,0), (3.5,1,0), (0.5,0,0),(1.5,0,0), (2.5,0,0), (3.5,0,0), (0.5,1,0), (1.5,1,0), (2.5,1,0), (3.5,1,0), (0.5,4,0), (1.5,4,0), (2.5,4,0), (3.5,4,0)]. Note that this set contains nonequilibrium points as well.

Figure 2 shows the controlled system's response to a step of amplitudes 3 and 4.

It is evident from the time histories that the off-equilibrium solution outperforms the conventional one. The conventional fuzzy controller cannot track the command with amplitude 4, while the off-equilibrium fuzzy controller tracks the command with fast settling time and little overshoot. Although the final state (4,0) belongs to the stability manifold, the conventional controller cannot reach it in response to a step command because its state crosses regions where $x_2 \gg 0$,which is too far from the equilibrium manifold. The off-equilibrium controller succeeds in tracking the command because it was designed to be stable in a larger region XU containing all of the state trajectories. From condition 1 we have that the convex combination of fuzzy controllers in the intermediate region retains stability.

Robustness to approximation error is also satisfied. In the calculation of course, the structured uncertainty matrices are zeroed for the dynamic matrix entries relative to linear functions. (In such a case even if the Jacobian matrix entry is different from zero, the approximation error results null.) That is, if the x_1 derivative were nonlinear, the approximation test would fail suggesting a necessity for a finer grid. (The Johansen[7] index in the proposed example depends mainly on the grid step $\Delta\{J_i, conv[(\bar{x}_i, \bar{u}_i), i \in J_i]\}$.)

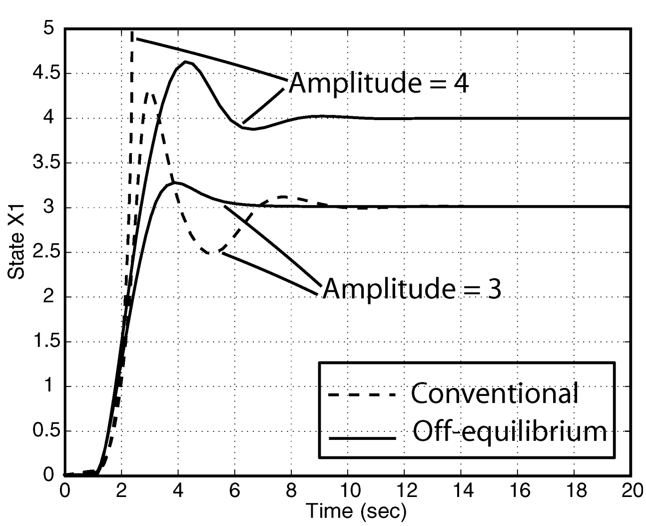

Fig. 2 Step response.

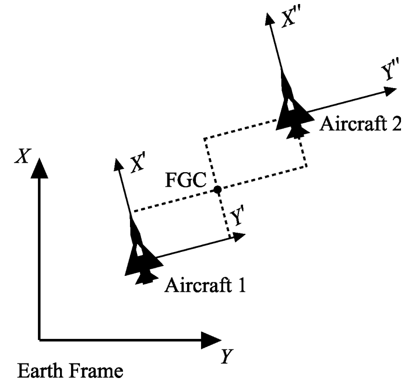

Fig. 3 Schematic of a two-ship formation.

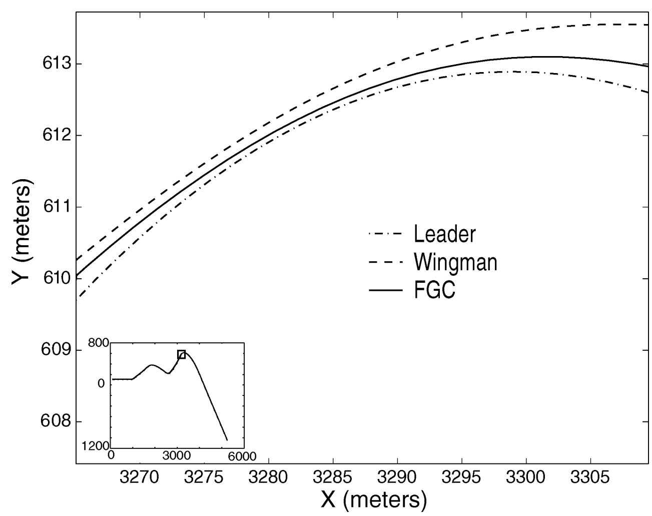

Fig. 4 Heading change tracking.

The second case study is a formation control problem for unmanned aircraft executing tight turns as depicted in Fig. 3.

The dynamics of the aircraft are based on a standard point-mass model with speed, flight-path angle, and heading angle as states, and thrust, bank angle, and load factor as inputs. The formation error is computed as distances of leader and wingman from the formation geometric center (FGC) defined in Ref. 8, where complete modeling and autopilot design are presented. The formation controller is limited to planar motion with a constant speed of 20 m/sec (Ref. 9), and the only variation on the heading angle $\chi \in (-\pi/4, \pi/4)$; this implies $XU = [-\pi, \pi] \times [-\pi, \pi]$. Nine operating points were used to form the grid, and for each point a standard LQ-servo was designed using a LMI procedure.[4] Performance comparison was made between TS fuzzy gain scheduling and a crisp scheduler.[6,10] The stability test described in the preceding section was successful, as well as the test for the approximation error ($\varepsilon = 7.969$, $\Delta = \pi/8$, $M = 19.6$). A comparison was made between the conventional controller (rigid gain scheduling with no mixing of gains) and the fuzzy gain-scheduled system (FGS). The validation and comparison were performed by testing for a sequence of tight turns for which the crisp scheduled controller becomes unstable because is more sensitive to high-frequency changes in the scheduling, while the fuzzy one does not. Results of the performance of the FGS are shown in Figs. 4 and 5.

In Fig. 4 we can see a detailed section of the trajectory indicating the closeness of the formation and the position of the aircraft relative to the FGC. Figure 5 shows the behavior of the X–Y distances with

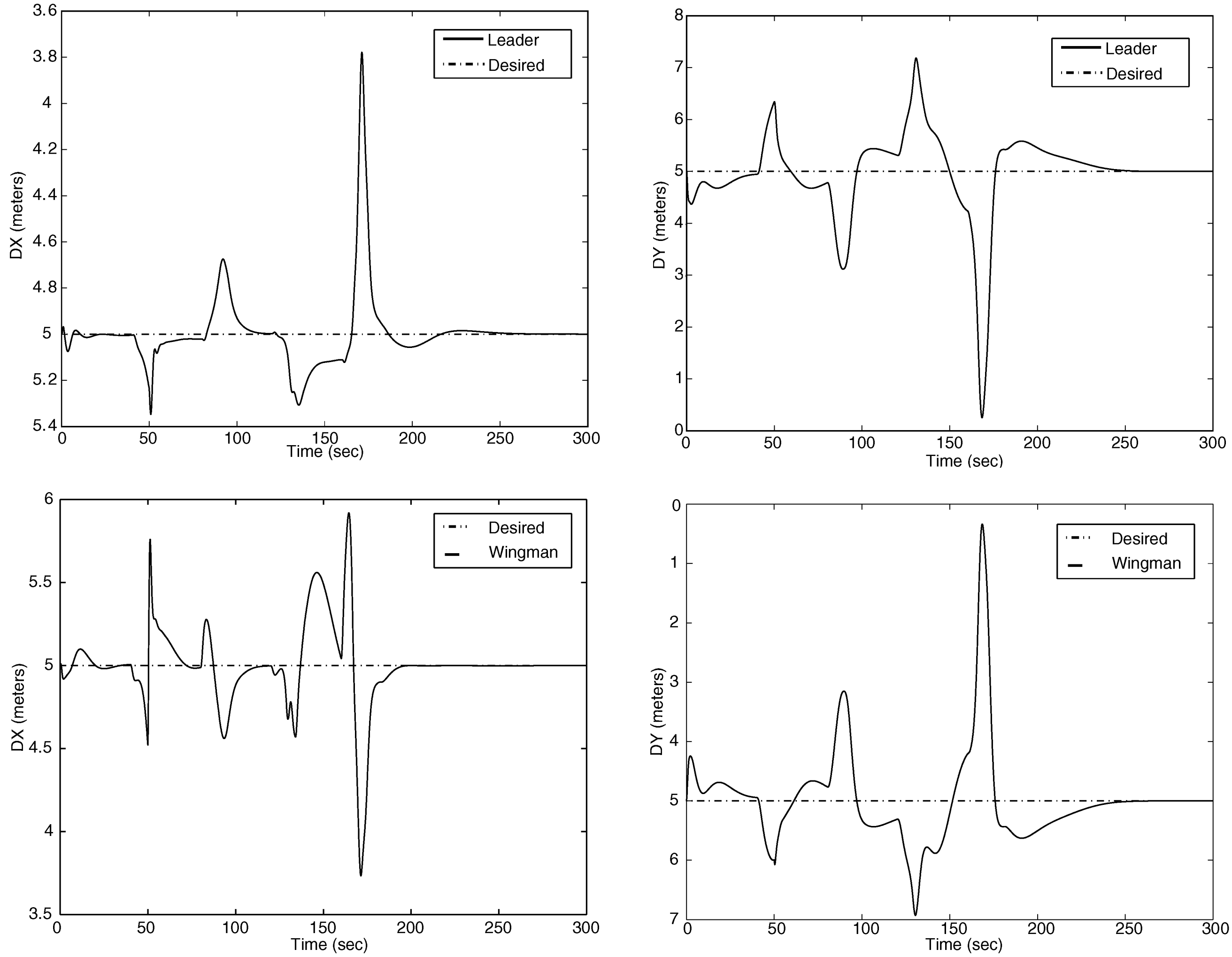

Fig. 5 Horizontal distances to FGC.

respect to the formation geometry center, with the nominal (desired) values set at 5 m for each component.

V. Conclusions

A new stability criterion for fuzzy gain-scheduled controllers was presented. The design procedure uses methods to solve linear matrix inequalities and differential inclusion techniques to guarantee stability of the closed-loop system within the desired state-space region. The procedure also provides the designer with information that can aid in grid redefinition, when a candidate grid is too sparse in critical regions of the state-control space. Two case studies demonstrate the method by comparing equilibrium and off-equilibrium fuzzy controllers and a fuzzy scheduler against a standard crisp scheduler.

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